Eigenvalue problem approach to the blind source separation: Optimization with a reference signal

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The eigenvalue problem approach to the blind source separation [L. Molgedey and H. G. Schuster, Phys. Rev. Lett. **72**, 3634 (1994)] is reinvestigated. The essential assumption is that the source signals should be statistically independent for the eigenvalue method to be applicable. When the source signals are correlated, unfortunately, this elegant approach faces a serious problem of optimization. We propose that by employing a reference signal in the separation procedure, the reconstructed signals that have an optimum minimum mismatch to the original sources can be obtained. The role and the criterion in choosing the reference signal will be extensively illustrated. Furthermore, the influences of nonzero correlation between different source signals, finite data length, and channel noises on signal separation will also be fully clarified.
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I. INTRODUCTION

Blind source separation has been an intriguing issue for a long time, partially due to its similarity to the human experience, e.g., the cocktail party effect [1]. It also has great application to various fields. The simplest case of the source separation problem occurs between two speakers whenever the mixture of their voices reaches two microphones and one wants to separate both sources such that each output channel registers only one voice [2]. Further examples, involving many sources and receivers, are the separation of odors in a mixture by an array of sensors and the parsing of the environment into different objects by our visual system [3]. Because of the complexity inherent in these problems, blind source separation has stood as an unsolved problem in history. The difficulty exhibited in such source separation could be described as follows. Assume that there are n source signals, $a_i(t)$ $(i=1,2,...,n)$, and that they satisfy the correlation relation $\langle a_i(t)a_j(t')\rangle = K_i(|t-t'|)\delta_{ij}$, which means that they are *statistically independent*. Meanwhile, assume that there are n detectors that can provide n detected signals $I_i(t)$ $(i=1,...,n)$. Thus the detected signals can be related to the source signals by $I_i(t)=\sum_{j=1}^{n} C_{ij}a_j(t)$, where C_{ij} are the mixing coefficients. Because the matrix C is generally not symmetric and the source strengths are not available, the total unknown number of variables is $n(n+1)$. On the other hand, there are only n detected signals available. The classical problem is *how one determines the coefficients C_{ij} and the source strengths $\lambda_i = K_i(0)$ from a measurement of $I_i(t)$*.

Unlike other methods using neural network approach [4–6], cumulants, and polyspectra [7–10], Molgedey and Schuster (MS) used time-delayed correlation to separate a mixture of independent signals [11]. Since there is no cost function or adaptation rule that may cause conceptual difficulty and ambiguity, this approach is physically straightforward and appealing. Molgedey and Schuster show that although the symmetric correlation matrix $\langle I_i(t)I_j(t)\rangle \equiv M_{ij}$ alone cannot solve the problem, the time-delayed correlation matrix $\langle I_i(t)I_j(t+\tau)\rangle \equiv \overline{M}_{ij}$ can provide the additional conditions for resolving the problem. In a more explicit form,

$$M_{ij}=\sum_l C_{il}C_{jl}\lambda_l, \quad \overline{M}_{ij}=\sum_l C_{il}C_{jl}\overline{\lambda_l}, \tag{1}$$

where C_{ij}, λ_i, and $\overline{\lambda_i}=K_i(\tau)$ are unknown. With Eq. (1), one can diagonalize M and $\overline{M}$ simultaneously, and this gives $C^{-1}M(C^T)^{-1}=\Lambda$ and $C^{-1}\overline{M}(C^T)^{-1}=\overline{\Lambda}$ where $\Lambda_{ij}=\lambda_i\delta_{ij}$ and $\overline{\Lambda}_{ij}=\overline{\lambda_i}\delta_{ij}$. Eventually, Eq. (1) leads to an eigenvalue problem,

$$(M\overline{M}^{-1})C=C(\Lambda\overline{\Lambda}^{-1}). \tag{2}$$

Usually, $M\overline{M}^{-1}$ is not symmetric. Because C is formed by the eigenvectors, the diagonal elements of C can be normalized to unity. Note that for simplicity, the signal is with zero mean, i.e., $\langle a_i(t)\rangle=0$. This zero-mean assumption will not affect the result because the essential message is included in the time-varying part. Equation (2) can be solved by the standard techniques of numerical linear algebra with high computational efficiency. As an impressive example, Molgedey and Schuster successfully separated the mixed sounds from two independently crying babies [11].

The blind source separation based on the MS scheme is rather novel, but there remains one critical parameter, i.e., the time delay τ, that must be clarified. It has been suggested that this problem is solvable simply by choosing the time delay that satisfied $K_i(0)K_j(\tau)=\lambda_i\overline{\lambda_j}\neq\overline{\lambda_i}\lambda_j=K_i(\tau)K_j(0)$ for all $i\neq j$ [11]. But this condition implies an already successful separation, and its validity is based on three conditions (separation conditions), i.e., (1) $\overline{M}$ is not a singular matrix, (2) the eigenvalues of $M\overline{M}^{-1}$ are nondegenerate, and (3) the eigenvalues of $M\overline{M}^{-1}$ should be real. As soon as the eigenvectors are deduced, Eq. (2) implies that after the separation, the mismatches for any τ should be the same and all are exactly equal to zero. This zero-mismatch property is remarkable since it means that the classical blind source separation has been perfectly solved. But, as will be shown below, if (i) the statistical independence of the source signals is not satisfied and (ii) the data length available is finite, then this zero-mismatch property disappears and the MS method may not be applicable. Hence, in practice, numerous τ's lead to separation, but which τ can result in an optimum is unknown. Since the MS method is a straightforward method with high computation efficiency, any solution to this problem requires further investigation.

As suggested by Molgedey and Schuster [11], it is also possible to introduce a cost function whose minima determine the mixing coefficient matrix. With such a cost function approach, Ehlers and Schuster detailed a two-step separation procedure that brings out the optimal solution for the separation problem of correlated mixed signals [12]. In this paper we will present another possible solution to the weakness in the MS method, namely, a separation procedure assisted with reference signal (SPARS). The merit of this SPARS approach is that the separation process is essentially an eigenvalue problem approach and thus physically appealing and straightforward. Meanwhile, the result of the separation procedure is reliable as supported by the perturbation analysis. In Sec. II it will be analytically shown that the zero-mismatch property does not hold in the case of finite source cross-correlation. However, in Sec. III we will show that with SPARS one can obtain a new handle on choosing the optimum time delay τ that can result in a minimal mismatch between the reconstructed signals and the original source signals. Two criteria for judging the goodness of the optimization will be proposed in Sec. IV. We define the numerical setting of simulation in Sec. V. In Sec. VI simple and known source signals are employed to illustrate the details of the SPARS method. The influence of finite data length on signal separation will also be clarified. In Sec. VII we will extend the analysis to complex time series to further test the applicability of the SPARS method. The effect of channel noise will be addressed in Sec. VIII. Finally, we summarize our work in Sec. IX as the conclusion.

II. INFLUENCE OF FINITE CROSS-CORRELATION ON THE MS SCHEME

Let us first give an exact formalism for the case of finite source cross-correlation for the MS scheme, in this case, the cross-correlation

$$\langle a_i(t)a_j(t)\rangle = Q_{ij}, \quad \langle a_i(t)a_j(t+\tau)\rangle = \overline{Q}_{ij}. \tag{3}$$

Thus the matrices M and $\overline{M}$ become $M=CQC^T$ and $\overline{M}=C\overline{Q}C^T$, respectively. Here, Q ($\overline{Q}$) is the matrix with elements Q_{ij} ($\overline{Q}_{ij}$). Now, Eq. (2) is replaced by

$$(M\overline{M}^{-1})C = C(Q\overline{Q}^{-1}). \tag{4}$$

We can set $Q\overline{Q}^{-1}=S\tilde{\Lambda}S^{-1}$, thus

$$(M\overline{M}^{-1})CS = CS(\tilde{\Lambda}). \tag{5}$$

By the eigenvalue $\tilde{\Lambda}$ and eigenmatrix CS, the reconstructed outputs A', a column vector formed by a_i', follow $A'=(CS)^{-1}I$. Since $I=CA$, the connection between the reconstructed and the source signals is

$$A' = S^{-1}A, \tag{6}$$

where A is the column vector formed by a_i. Note that by definition S (S^{-1}) is τ dependent, hence, the reconstructed signals will be different from the original source signals for different delay time τ. Now, in contrast to the case of zero cross-correlation, intrinsic finite cross-correlation inevitably causes the imperfection in the separation, which simply implies that the solution could not be uniquely determined. This deeply reflects the nature of the blind source separation in the inverse problem we have to face. In general, nonuniqueness in solution is inevitable in handling the inverse problem [13].

Let us further explore this imperfect feature by a perturbation analysis. For a small nonzero cross-correlation, the correlation may follow

$$\langle a_i(t)a_j(t')\rangle = K_i(|t-t'|)\delta_{ij} + \varepsilon L_{ij}(|t-t'|), \tag{7}$$

where ε is used to denote the strength of the source cross-correlation strength. Again, let us set $\langle a_i(t)\rangle = 0$. We define $\lambda_i(0)=\underline{\lambda}_i$, $\lambda_i(\tau)=\lambda_i$; $L_{ij}(0)=l_{ij}$, $L_{ij}(\tau)=\overline{l}_{ij}$, $(i\neq j)$, and $l_{ii}=0=\overline{l}_{ii}$. The matrices $M'_{ij}=\langle I'_i(t)I'_j(t)\rangle$ and $\overline{M}'_{ij}=\langle I'_i(t)I'_j(t+\tau)\rangle$ can be deduced. They are $M'=C\Lambda C^T+\varepsilon CLC^T$ and $\overline{M}'=C\overline{\Lambda}C^T+\varepsilon C\overline{L}C^T$. Thus

$$C^{-1}M'(C^T)^{-1} = \Lambda + \varepsilon L, \quad C^{-1}\overline{M}'(C^T)^{-1} = \overline{\Lambda} + \varepsilon\overline{L}. \tag{8}$$

We invert the second part of Eq. (8), i.e.,

$$C^T(\overline{M}')^{-1}C = \frac{1}{\overline{\Lambda}+\varepsilon\overline{L}} \approx \overline{\Lambda}^{-1}(U - \varepsilon\overline{L\Lambda}^{-1}), \tag{9}$$

where U is the unit matrix. By Eqs. (8) and (9) and after some algebraic calculations, it can be deduced that

$$M'\overline{M}'^{-1}C = C\Lambda\overline{\Lambda}^{-1}[U + \varepsilon(\Lambda^{-1}\overline{\Lambda}L\overline{\Lambda}^{-1} - \overline{L\Lambda}^{-1}) + O(\varepsilon^2,\tau)]. \tag{10}$$

One can see that a suitable choice of τ can lead to $(\Lambda^{-1}\overline{\Lambda}L\overline{\Lambda}^{-1} - \overline{L\Lambda}^{-1}) \approx 0$, and thus an $O(\varepsilon^2)$ approximation. This change of mismatch is caused by τ, and it clearly shows that the time delay controls the separation performance.

III. FORMALISM OF THE REFERENCE SIGNAL APPROACH

Next, we present one possible approach to resolving the above-mentioned difficulty in the MS method. Our approach is to introduce one *virtual* channel where a chosen reference signal is embedded. For illustration and comparison, we first simply assume the source independence still holds. We now have new $(n+1)$ ''virtually detected'' signals as $I_i'(t)=I_i(t)+b_i a_{\rm ref}(t)$, $i=1,2,...,n$ and $I_{n+1}'(t)=a_{\rm ref}(t)$ where $a_{\rm ref}(t)$ is the reference signal and b_i is the mixing coefficients, which can be set as 1 for simplicity. We choose $\alpha\sin(wt)$ as the reference signal, and thus $\langle a_{\rm ref}(t)a_{\rm ref}(t+\tau)\rangle=(\alpha^2/2)\cos(w\tau)$. By setting B as the column vector formed by b_i, the individual correlation functions can be written as

$$\langle b_i a_{\rm ref}(t)a_{\rm ref}(t+\tau)\rangle=\frac{\alpha^2}{2}\cos(w\tau)b_i\rightarrow\frac{\alpha^2}{2}\cos(w\tau)B, \tag{11}$$

$$\langle a_{\rm ref}(t)b_j a_{\rm ref}(t+\tau)\rangle=\frac{\alpha^2}{2}\cos(w\tau)b_j\rightarrow\frac{\alpha^2}{2}\cos(w\tau)B^T, \tag{12}$$

$$\langle b_i a_{\rm ref}(t)b_j a_{\rm ref}(t+\tau)\rangle=\frac{\alpha^2}{2}\cos(w\tau)b_i b_j \rightarrow\frac{\alpha^2}{2}\cos(w\tau)BB^T, \tag{13}$$

where the correlation of reference signals has been applied. For simplicity, here we also assume $\langle a_{\rm ref}(t)a_i(t+\tau)\rangle=0$ for all i. This also means that we consider a *noncorrelated* reference signal first.

In the same way as the eigenvalue approach [11], the contribution of the reference signal to SPARS can be analyzed. The matrices $M_{ij}'=\langle I_i'(t)I_j'(t)\rangle$ and $\overline{M}_{ij}'=\langle I_i'(t)I_j'(t+\tau)\rangle$ are

$$M_{ij}'\rightarrow\begin{pmatrix}\langle I_i(t)I_j(t)\rangle+\langle b_i a_{\rm ref}(t)b_j a_{\rm ref}(t)\rangle & \langle b_i a_{\rm ref}(t)a_{\rm ref}(t)\rangle\\ \langle a_{\rm ref}(t)b_j a_{\rm ref}(t)\rangle & \langle a_{\rm ref}(t)a_{\rm ref}(t)\rangle\end{pmatrix},$$

i.e.,

$$M'=\begin{pmatrix} M+\frac{\alpha^2}{2}BB^T & \frac{\alpha^2}{2}B\\ \frac{\alpha^2}{2}B^T & \frac{\alpha^2}{2}\end{pmatrix}, \tag{14}$$

and

$$\overline{M}_{ij}'\rightarrow\begin{pmatrix}\langle I_i(t)I_j(t+\tau)\rangle+\langle b_i a_{\rm ref}(t)b_j a_{\rm ref}(t+\tau)\rangle & \langle b_i a_{\rm ref}(t)a_{\rm ref}(t+\tau)\rangle\\ \langle a_{\rm ref}(t)b_j a_{\rm ref}(t+\tau)\rangle & \langle a_{\rm ref}(t)a_{\rm ref}(t+\tau)\rangle\end{pmatrix},$$

i.e.,

$$\overline{M}'=\begin{pmatrix}\overline{M}+\frac{\alpha^2}{2}\cos(w\tau)BB^T & \frac{\alpha^2}{2}\cos(w\tau)B\\ \frac{\alpha^2}{2}\cos(w\tau)B^T & \frac{\alpha^2}{2}\cos(w\tau)\end{pmatrix}. \tag{15}$$

The inverse matrix of $\overline{M}'$ thus follows:

$$\overline{M}'^{-1}=\begin{pmatrix}\overline{M}^{-1} & -\overline{M}^{-1}B\\ -B^T\overline{M}^{-1} & \left(\frac{\alpha^2}{2}\cos(w\tau)\right)^{-1}\left(1+\frac{\alpha^2}{2}\cos(w\tau)B^T\overline{M}^{-1}B\right)\end{pmatrix}. \tag{16}$$

With Eqs. (14) and (16), we obtain one of the key results,

$$M'\overline{M}'^{-1}=\begin{pmatrix} M\overline{M}^{-1} & \{[\cos(w\tau)]^{-1}-M\overline{M}^{-1}\}B\\ 0 & [\cos(w\tau)]^{-1}\end{pmatrix}. \tag{17}$$

Thus, in the case of *zero* source cross-correlation and a *noncorrelated* reference signal, the separation performance of SPARS is exactly the same as that of the MS scheme. It can be shown that

$$M'\overline{M}'^{-1}C'=C'\Lambda',$$

$$\Lambda'=\begin{pmatrix}\Lambda\overline{\Lambda}^{-1} & 0\\ 0 & [\cos(w\tau)]^{-1}\end{pmatrix},$$

$$C'=\begin{pmatrix} C & B\\ 0 & 1\end{pmatrix}. \tag{18}$$

In other words, after solving the eigenvectors, the reconstructed signals are

$$C'^{-1}I' = \begin{pmatrix} A \\ a_{\mathrm{ref}} \end{pmatrix}. \tag{19}$$

For the case where the source cross-correlation is *not zero*, but still with a noncorrelated reference signal, it can be shown that

$$M'\overline{M}'^{-1}C' = C'\Lambda',$$

$$\Lambda' = \begin{pmatrix} \Lambda\overline{\Lambda}^{-1} & 0 \\ 0 & [\cos(w\tau)]^{-1} \end{pmatrix}, \tag{20}$$

$$C' = \begin{pmatrix} CS & B \\ 0 & 1 \end{pmatrix},$$

and the reconstructed signals are

$$C'^{-1}I' = \begin{pmatrix} S^{-1}A \\ a_{\mathrm{ref}} \end{pmatrix}, \tag{21}$$

after following exactly the same calculation procedure shown above.

IV. OPTIMIZATION CRITERIA AND THE RECIPE FOR REFERENCE SEARCHING

It seems that there is no need for a noncorrelated reference signal since SPARS is exactly the same as the MS scheme in mathematics. However, it should be kept in mind that the MS scheme has a serious optimization problem. In contrast, for the SPARS method, we know the reference signal. By the well-chosen reference signal, we can deduce an optimal time delay by minimal mismatch between the reconstructed and the original reference signals. This provides advantages beyond the MS scheme.

Intuitively, a successful separation should bring one of the reconstructed signals close to the reference signal with a minimum mismatch and the eigenvalue should be close to $[\cos(w\tau)]^{-1}$. For a noncorrelated reference signal, we know that with the chosen time delay, $a'_{\mathrm{ref}} \rightarrow a_{\mathrm{ref}}$ and $A' \rightarrow S^{-1}A$, after solving the eigenvector. Is it possible that $A' \rightarrow S^{-1}A \rightarrow A + O(\varepsilon^2)$ for a chosen τ? This is impossible for a noncorrelated reference signal as shown above. Nevertheless, the results of noncorrelated reference signals are important since they indicate that the reference signal should relate to the detected signals. On the other hand, the perturbation analysis suggests that with a correlated reference signal, it is possible to have such an improvement of source separation. Thus we take a reference signal weakly correlated to the detected signals. We also know that if the separation could not lead to a minimum mismatch for the reference signal, the success of separation is questionable. This observation gives us two criteria for getting an optimized signal separation. One is based on the mismatch between the reference signal and its reconstructed counterpart. Another one can be established in terms of the eigenvalue, i.e., the optimization should be achieved when the error function, which is defined as

$$E = \left| \frac{1}{\cos(w\tau)} - \left(\frac{\lambda_{\mathrm{ref}}}{\overline{\lambda}_{\mathrm{ref}}} \right) \right|, \tag{22}$$

can be minimized. Here $(\lambda_{\mathrm{ref}}/\overline{\lambda}_{\mathrm{ref}})$ are the corresponding eigenvalues of $M'\overline{M}'^{-1}$ for the reference signal.

Obviously, the reference signal is crucial to SPARS. It is urgent to set up an efficient procedure to derive the reference signal. Since we have shown that a noncorrelated reference signal could not provide further improvement, we should take a sine signal with the least correlation to the detected signal. This is to avail ourselves of the advantage suggested by the perturbation analysis. Also, as suggested by the analytical result, we have numerically verified that it is not wise to choose the frequency components outside the spectrum range of the detected signals, since they usually do not pass the double check criteria listed below. Thus, practically, to set the range of frequency for searching for the reference signal, it would be useful to take a power spectrum on the detected signals first. The recipe for the SPARS method is as follows.

(1) Perform a power spectrum analysis on the detected signals to determine the spectrum region.

(2) Take a sine signal whose frequency is within the region and calculate its cross-correlation to the detected signals.

(3) Choose the least-correlated sine signal as the reference.

(4) After the separation, run a double check on the concurrence of minima in the reference mismatch and error function to ensure the correctness of the chosen delay time.

On the other hand, as may have been expected from Eq. (18), where the reference amplitude α vanishes, the amplitude of the reference signal is found to be not so important. Thus one can simply choose the reference amplitude with the same order as the detected signal. We will provide numerical support in the following sections.

V. NUMERICAL SETTINGS OF SIMULATION EXPLORATION

Before presenting the numerical results, let us first quantify the degree of success for a separation of n source signals. We define Δ_j as the difference between the maximum and the minimum of the original signal $a_j(t)$. To evaluate the mismatch per unit time interval, we perform a long time average of the absolute difference between the separated signal $a'_i(t)$ and the original signal $a_j(t)$, i.e., $\langle |a'_i(t) - a_j(t)| \rangle = (1/T)\int_0^T |a'_i(t) - a_j(t)| dt$, where T should be long enough. The minimum mismatch of the reconstructed signals $a'_i(t)$, named δ_i, is defined as the minimum of $\delta_{i,j} = \langle |a'_i(t) - a_j(t)| \rangle / \Delta_j$, where $j = 1,2,...,n$. This method of definition is necessary because in practice we need to compare each reconstructed signal with all source signals to identify the correspondence. Finally, we can quantify the average degree of mismatch for the whole separation in a unit time interval by defining $\delta = (1/n)\sum_{i=1}^{n}\delta_i$ where a normalized intensity scale has been used.

Next, we define some notations used in the paper. The source mismatch δ is for the sources we hope to separate *based on the MS scheme*. On the other hand, for SPARS, we

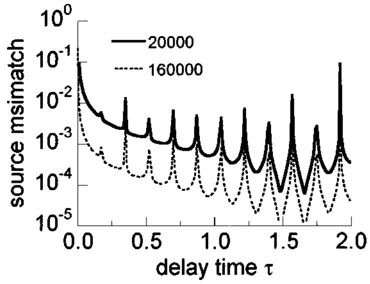

FIG. 1. Source mismatch δ versus the delay time τ for different data lengths.

use a different kind of symbol δ_s for the source mismatch which here means only that the sources we hope to separate are employed to calculated the mismatch. The mismatch of reference signal is not included. In the meantime, the mismatch between the reference signal and the reconstructed signal, which is close to the reference signal, is defined as δ_r.

For the numerical testing presented in this paper, the mixing coefficient matrix is given as

$$C=\begin{pmatrix} 1.0 & 0.9 \\ 0.7 & 1.0 \end{pmatrix},$$

like that in Ref. [11], and we also set $B=1$. We note that a different choice of mixing coefficient matrix does not change the general features reported below. In simulation, we employed a fourth-order Runge-Kutta algorithm to integrate the differential equations. Usually, the time step is 0.01 and the data length for evaluation is, on average, 20 000. All data sets used here are taken after the transient, and the means of the data sets have been shifted to zero.

VI. NUMERICAL EXPLORATION WITH SIMPLE, KNOWN SIGNALS

A. The influence of data length

The zero-mismatch property in the MS scheme is a remarkable achievement. As shown above, a violation of statistical independence between the source signals leads to a breakdown of this unique property. Here, we will demonstrate that data length also affects this property greatly.

For simple illustration, let us take the source signals to be $a_1(t)=\sin(17t)$ and $a_2(t)=\sin(19t)$. Thus, in principle, there is no cross-correlation between the source signals. For comparison, we use two different data lengths, 20 000 and 160 000. As shown in Fig. 1, the separation performance changes as τ is varied while a longer data length reduces the mismatch. One can expect that unless the separation condition is violated, the zero-mismatch property will be met when the data length is infinite. Here the divergence originates from det $\underline{M}=0$ as $\tau=(n+1/2)\pi/\omega$, where $\omega=17$ and 19, and $n=1,2,\ldots$. We should emphasize that since the available data length is always finite, different τ's lead to different separation performance. The effect of finite data length is also inevitable in the SPARS method. However, as will be shown below, one can still deduce an optimal time delay for source separation in SPARS, and thus the influence of finite data length can be effectively removed.

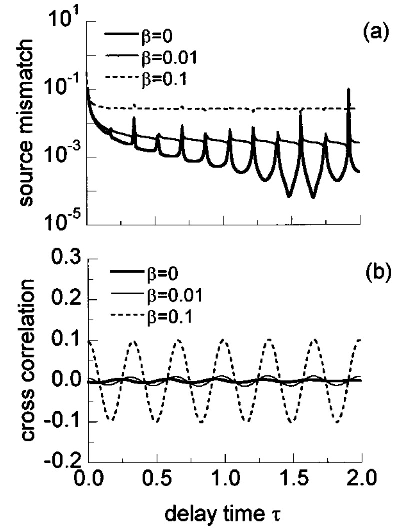

FIG. 2. (a) Source mismatch δ versus delay time τ for two source signals whose forms are sine. (b) Cross-correlation of source signals.

B. The influence of cross-correlation

To illustrate the influence of cross-correlation on signal separation, let us take the source signals to be $a_1(t)=\sin(17t)+\beta\,\sin(19t)$ and $a_2(t)=\sin(19t)$, and choose $a_{\mathrm{ref}}(t)=\sin(23t)$. The signals to be separated are $I_1'(t)=I_1(t)+a_{\mathrm{ref}}(t)$, $I_2'(t)=I_2(t)+a_{\mathrm{ref}}(t)$, and $I_3'(t)=a_{\mathrm{ref}}(\underline{t})$. We construct the matrix $M_{ij}=\langle I_i'(t)I_j'(t)\rangle$ and $M_{ij}=\langle I_i'(t)I_j'(t+\tau)\rangle$, where $i,j=1,2,3$, and solve the eigenvectors of the matrix MM^{-1}. The cross-correlation between the sources is

$$\frac{\langle a_1(t)a_2(t+\tau)\rangle}{\sqrt{\langle a_1^2(t)\rangle}\sqrt{\langle a_2^2(t)\rangle}}=\frac{\beta}{\sqrt{1+\beta}}\cos(w\tau)$$

and $w=19$. By setting $\beta=0.0$, 0.01, and 0.1, the influence of nonzero cross-correlation can be explored. Again, we first present the source mismatch obtained by the MS scheme in Fig. 2(a). The cross-correlation is presented in Fig. 2(b). One can see that as the statistical independence is improved (cross-correlation is small), there is more of a chance to achieve a better separation. This feature is also true for SPARS, as shown in Fig. 3. Unlike in Fig. 2(a), there appear more divergence (discontinuity) points for SPARS due to the introduction of the reference signal. On the other hand, referring to Figs. 3(b) and 3(c), the reference mismatch and the error function remain unchanged as the cross-source correlation β varies. This is because the cross-correlation between source and reference signals is zero.

Next, we consider the situation in which the reference signal is correlated to the source signals. Let us take a different reference signal, $a_{\mathrm{ref}}(t)=\sin(11t)$, to show the generality. We also take $a_1(t)=\sin(17t)+\beta\,\sin(11t)$ and $a_2(t)=\sin(19t)$. Again, we take $\beta=0.0$, 0.01, and 0.1 to explore the influence of this finite correlation. The results are shown in Fig. 4. One can see that a strong cross-correlation between the reference and source signals results in a failure of signal separation. Please note that the variation feature remains un-

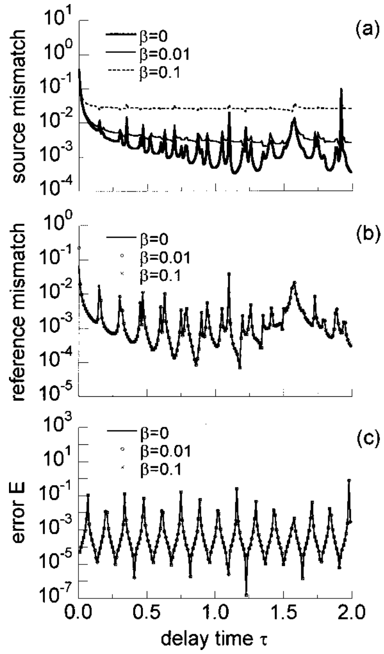

FIG. 3. (a) Source mismatch δ_s versus delay time τ for two source sine signals in which different correlation coefficients β are used. (b) Reference mismatches δ_r. (c) Error function E. The original source signals are the same as those used in Fig. 2.

changed in the error function. Thus the concurrence of the local minima at the same τ for the reference mismatch and the error function completely disappears when the reference signal is strongly correlated to the source signals. This is a useful result. We can use this property as a double check of the correctness of separation performance.

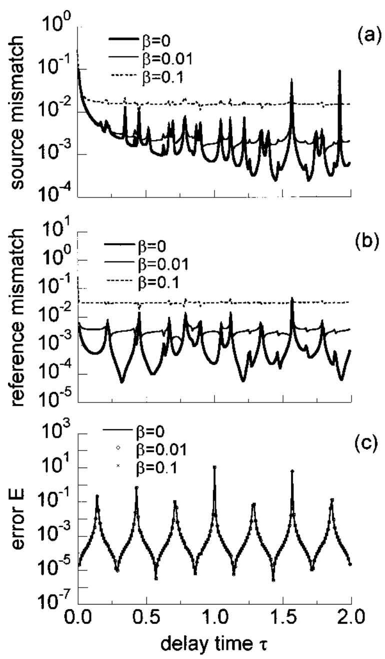

FIG. 4. (a) Source mismatch δ_s versus delay time τ for two source sine signals with different correlation coefficient β. (b) Reference mismatches δ_r. (c) Error function E.

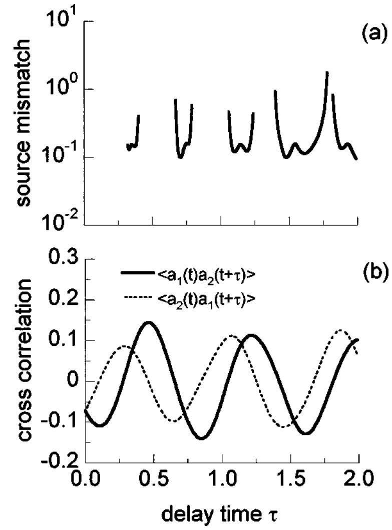

FIG. 5. (a) Source mismatch δ versus delay time τ for the mixed chaotic signals. (b) Cross-correlation of source signals.

VII. NUMERICAL EXPLORATION WITH COMPLEX SIGNALS

A. Separation of mixed chaotic signals

In this section we will treat complex time series to give a further illustration of SPARS. Instead of the sound signals used in Ref. [11], here we take a chaotic signal [14] as a notable extension. These signals are deterministic, but their spectra are broadband which could not be separated simply with a filter.

We first choose $a_1(t)$ and $a_2(t)$ the z components of the Lorenz model with different values of r, i.e., $dx/dt = 10.0(x-y)$, $dy/dt = rx - y - xz$, $dz/dt = xy - 2.66z$, where $r = 28.0$ and 29.0, respectively. As shown in Fig. 5(a), there are gaps in the line and this indicates that the mismatch could not be evaluated in some regions. This is due to the fact that the separation conditions are violated—mainly, the eigenvalues of $M\bar{M}^{-1}$ are not real. This surely shows that separation no longer works for any arbitrary τ. Furthermore, the mismatch δ varied with time delay τ. Again, this clearly shows that zero mismatch does not hold in this case and the reason is the cross-correlation is not zero as shown in Fig. 5(b). For the SPARS method, we employ the same data set used in Fig. 5(a) and mingle the detected signals $I_1(t)$ and $I_2(t)$ with a reference signal $a_{\mathrm{ref}}(t) = 31\sin(17t)$. Referring to Fig. 6, one can see that the separation performance of the MS scheme is faithfully reproduced by SPARS at the local minima. The location of the time delay at the local minima for all three kinds of mismatch, i.e., δ, δ_s, and δ_r, and the error function E are closely matched. It can be seen that the values of δ and δ_s are almost the same when the time delays τ are located at the minima of the error function. On the other hand, the source mismatches δ and δ_s show different variation behavior as the τ moves away from the minima. In this case, the performance of SPARS is comparable to that of the MS scheme at the local minima. Let us examine more closely the role played by the reference signal, since in practice this is the only known ''source'' we have. The concurrence of local minima in E and δ_r at the same τ is expected, as analytically discussed in Sec. II. By this concurrence pat-

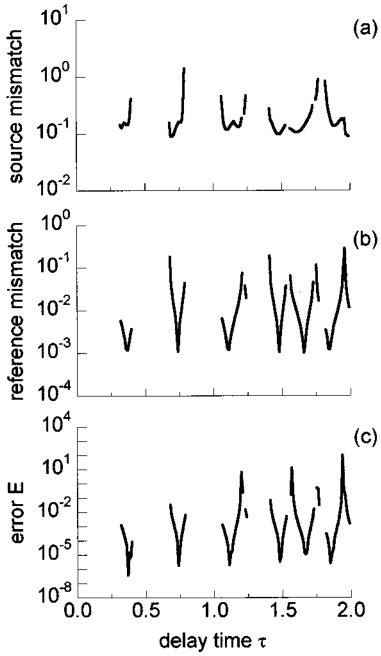

FIG. 6. (a) Source mismatch δ_s versus delay time τ for the data set used in Fig. 5 with reference signal $31\sin(17t)$. (b) Reference mismatches δ_r. (c) Error function E. The data set is the same as that used in Fig. 2.

tern we can deduce the optimal time delay for the signal separation. As shown in Figs. 5 and 6, this delay time τ also leads to an optimum separation, since the source mismatch is close to the minimum.

Let us consider another example where the source cross-correlation is smaller. This time we consider a separation of two mixed signals, one from the z component of the Lorenz chaos where $r=28.0$ and the other from the z component of the Rossler chaos, $dx/dt=-(y+z)$, $dy/dt=x+0.2y$, $dz/dt=0.4+(x-b)z$ with $b=5.7$. We first present the result derived based on the MS scheme and the statistical independence check in Fig. 7. One can see that the cross-correlation is much smaller in comparison with the above example. However, different τ's still lead to different separation performances and the variation in δ can be significant. Again, the zero-mismatch property does not hold. Next, we present the SPARS analysis. We take the reference signal to be $23\sin(17t)$. As shown in Fig. 8, the location of time delays for the local minima of δ_s, δ_r, and E are almost the same. This is further typical numerical evidence of the concurrence feature. Again, this emphasizes the advantage of SPARS in dealing with finite cross-correlation. Meanwhile, the ''global'' minimum of the error function, labeled by a down arrow as shown in Fig. 8, provides us with a very good source mismatch while the local minima of the reference mismatch are almost the same and do not provide us with more useful information.

B. Further investigation of the location of minimum

The exact location of the minimum influences the estimation of time delay and the separation performance. For the error function, the location of the minimum depends on the distribution of the numerically calculated eigenvalue. On the other hand, for the reference mismatch, the location depends

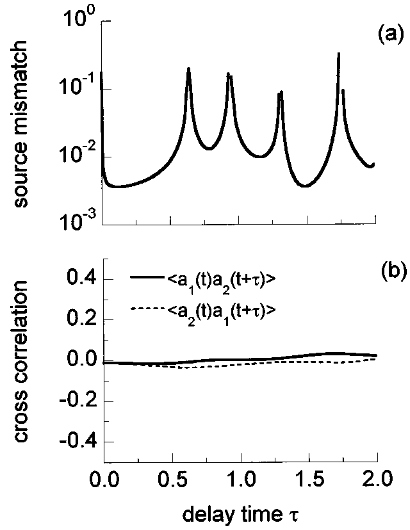

FIG. 7. (a) Source mismatch δ versus delay time τ for the mixed chaotic signals. (b) Cross-correlation of source signals.

on the reconstructed C_{3j} $(j=1,2,3)$ in the case of mixing two source signals. The eigenvalue will be calculated first and then the elements of the eigenvector matrix C_{ij}. Thus it is expected that the error function is more robust as the reference signal changes. This matches numerical results.

The exact location of a minimum depends on the resolution of τ and a higher resolution in τ gives a better estimation of the minimum. We term the minimum with a better estimation the ''global'' minimum. However, we should address this point more carefully, since the error function E is the absolute value of the difference between the exact and the numerically calculated eigenvalues. The distribution of the numerically calculated eigenvalue becomes critical. Mathematically, if the reference signal is perfectly statistically

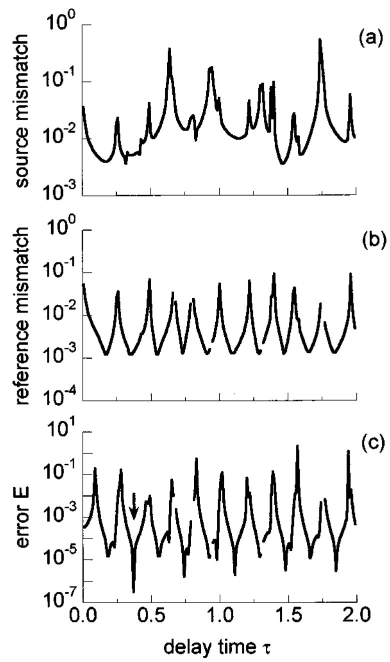

FIG. 8. (a) Source mismatch δ_s versus delay time τ for data set used in Fig. 7 with reference signal $23\sin(17t)$. (b) Reference mismatches δ_r. (c) Error function E.

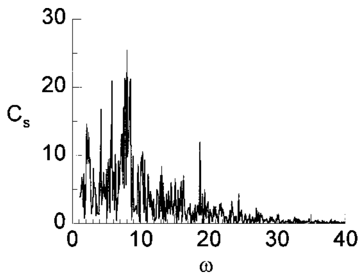

FIG. 9. Cross-correlation between source signals and reference signal versus frequency of the reference signal.

independent of the source signals, then, after solving the eigenvalue, the corresponding eigenvalue of the reference signal should be exactly equal to $1/\cos(\omega\tau)$. However, in our recipe, the reference signal should be weakly correlated to the source signals. Thus, one can expect the error function E to be close to zero. It is numerically found that the error function can be close to zero and, furthermore, $\cos(\omega\tau)\approx 1$, as shown in Fig. 8(c). Thus the minimum of the error function E occurs at $\tau = k\pi/\omega$ $(k=1,2,...)$. Although one can deduce the location of the minimum for the error function E, the location of the minimum for the reference mismatch remains undetermined.

C. Practical deduction of the reference signal

We should note that an arbitrarily chosen reference signal does not imply that it has to be a good reference signal. Here, we illustrate the influence of the reference signal in the case of complex time series. First, let us evaluate the correlation sum $C_s = |\langle I_1(t) a_{\rm ref}(t)\rangle| + |\langle I_2(t) a_{\rm ref}(t)\rangle|$, where $a_{\rm ref}(t) = \alpha \sin(wt)$ and $\alpha = 23$, for different w and the source signals are the same as those used in Fig. 7. As a typical example, let us address the separation of two different frequencies. As shown in Fig. 9, the correlation sum has a higher value at $w=8$ and a lower value at $w=17$. For these cases, the source mismatches are significantly different. One can see that the frequency component with least correlation to the source signals has a better separation performance, as shown in Fig. 10. Furthermore, in the case of low correlation, the concurrence feature of local minima at the same τ can be identified while this identification is difficult for the high correlation.

It is important to examine more closely the influence of the amplitude α. Let us still use the above example as an illustration. Let us first look at Fig. 11, where the frequency $w=17$ is a good reference signal. As expected from the analytical result, the source mismatch remains the same for different values of α. However, the reference amplitude affects reference mismatch when α is small. In contrast, there is almost no change in the pattern of the error function E. This again gives us one more numerical support and the concurrence of local minima at the same τ is a nice criterion. It is worthwhile to compare the case of $w=8$, which is not a good reference signal. As shown by Fig. 12, the feature shown in Fig. 11 remains: the amplitude α is not a critical parameter.

Please note that the reference signal with multiple frequencies can also be used. Let us take $a_{\rm ref}(t) = \alpha_1 \sin(w_1 t) + \alpha_2 \sin(w_2 t)$ as an example. In such a case, the $[\cos(w\tau)]^{-1}$ in the error function should be replaced by $(\alpha_1^2 + \alpha_2^2)/[\alpha_1^2 \cos(w_1\tau) + \alpha_2^2 \cos(w_2\tau)]$, which is more complicated. But, the silent feature of the error function, i.e., independence of the reference amplitude α, disappears, as shown in Fig. 13, where two different α_2 are presented and $w_1 = 17$ and $w_2 = 12$.

It is also possible to use multiple reference signal channels. Again, the analysis is more complicated and a separate error function is needed for every additional reference chan-

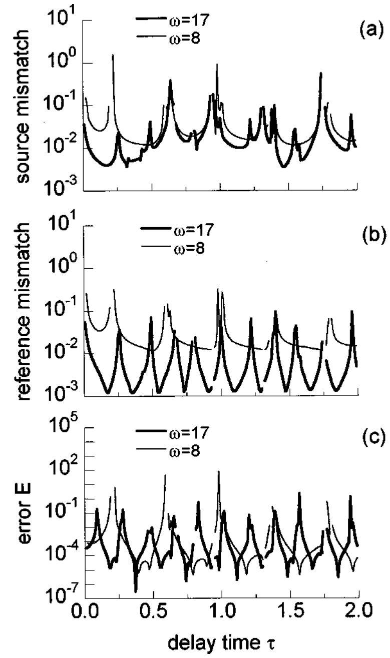

FIG. 10. (a) Source mismatch, (b) reference mismatch, and (c) error function for the mixed signals used in Fig. 7 with different reference signals.

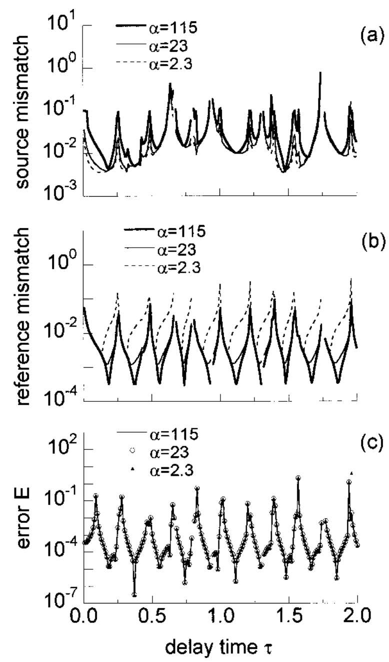

FIG. 11. (a) Source mismatch, (b) reference mismatch, (c) error function for the mixed signals used in Fig. 7 with different reference amplitude; here the reference is $\alpha \sin(17t)$ and α is the amplitude.

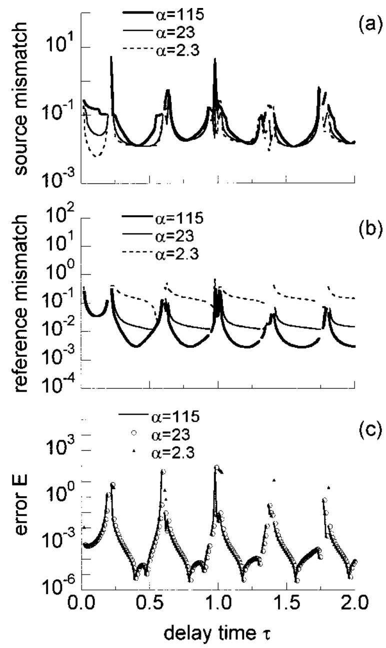

FIG. 12. (a) Source mismatch, (b) reference mismatch, (c) error function for the mixed signals used in Fig. 7 with different reference amplitude; here the reference is α sin(8t) and α is the amplitude.

nel. From numerical simulations, we learn that this extension offers no additional advantages, as shown in Fig. 14, where the separation performances of the MS scheme, SPARS with one reference channel, and SPARS with two reference channels can be seen.

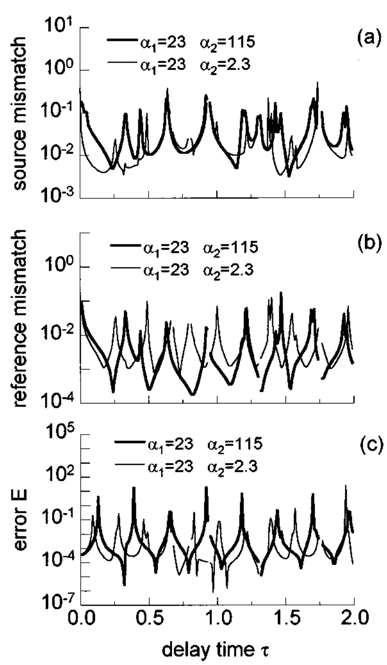

FIG. 13. (a) Source mismatch, (b) reference mismatch, and (c) error function for the mixed signals used in Fig. 7 with a reference signal α_1 sin$(17t)+\alpha_2$ sin$(12t)$ whose α_1 and α_2 are different.

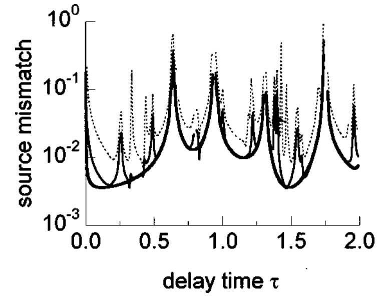

FIG. 14. Data set is the same as Fig. 7. Bold solid line is the performance of the MS scheme. Solid line is the performance of the SPARS method with one reference channel. Dashed line is the performance of the SPARS method with two reference channels.

VIII. INFLUENCE OF CHANNEL NOISE ON SIGNAL SEPARATION

In signal detection, channel noises are unavoidably embedded, and the influence on signal separation should be addressed. Without restriction, we can take the detected signals as $I_i'(t)=I_i(t)+\eta_i$, where η_i is the added channel noise at the ith detector. To simplify the discussion, we assume the sources are statistically independent and the channel noises are colored, i.e.,

$$\langle \eta_i(t)\eta_j(t+\tau)\rangle=\frac{D}{\tau_c}\exp\left(\frac{-\tau}{\tau_c}\right)\delta_{ij}=Df(\tau)\delta_{ij}, \quad (23)$$

where D denotes the noise intensity, τ_c is the correlation time, and $f(\tau)=(1/\tau_c)\exp[-\tau/\tau_c]$. It is reasonable to assume the noises are not relative to the sources such that

$$\langle \eta_i(t)a_j(t')\rangle=\langle a_i(t)\eta_j(t')\rangle=0. \quad (24)$$

Next, with channel noise and the new matrices $M'_{ij}=\langle I_i'(t)I_j'(t)\rangle$ and $\bar{M}'_{ij}=\langle I_i'(t)I_j'(t+\tau)\rangle$ it follows that

$$M'=M+Df(0)U, \quad \bar{M}'=\bar{M}+Df(\tau)U. \quad (25)$$

Therefore, $M'-Df(0)U=C\Lambda C^T$ and $\bar{M}'-Df(\tau)U=C\bar{\Lambda}C^T$, whose inverse is $[\bar{M}'-Df(\tau)U]^{-1}=(C^T)^{-1}\bar{\Lambda}^{-1}C^{-1}$. Thus

$$[M'-Df(0)U][\bar{M}'-Df(\tau)U]^{-1}C=C(\Lambda\bar{\Lambda}^{-1}). \quad (26)$$

Furthermore,

$$[\bar{M}'-Df(\tau)U]^{-1}=\frac{1}{\bar{M}'-Df(\tau)U}=\bar{M}'^{-1}[U+Df(t)\bar{M}'^{-1}+O(D^2)]. \quad (27)$$

After some algebra, it can be shown that

$$M'\bar{M}'^{-1}C=C(\Lambda\bar{\Lambda}^{-1})(U+D\{\bar{\Lambda}\Lambda^{-1}C^{-1}(C^T)^{-1}\times[f(0)-f(\tau)]\bar{\Lambda}^{-1}\}+O(D^2)). \quad (28)$$

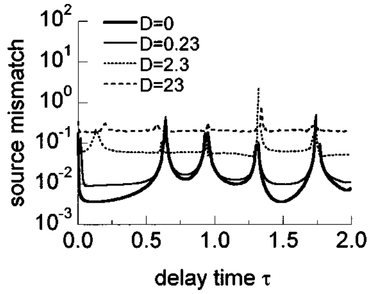

FIG. 15. Source mismatch versus delay time obtained based on the MS scheme with different detective noise level *D*. Data set is the same with Fig. 7.

Again, one can see that if the noise strength D is too strong, the MS scheme will fail, and so does the SPARS method. Therefore low-noise detectors are needed to ensure the reliability of signal separation.

Let us use the original data set of Fig. 7 with an addition of noise. Instead of colored noise, we take the noises as being a randomly, independently, and uniformly generated form $[-D,D]$. For comparison, we take $D=0.23$, 2.3, and 23, which are roughly 1%, 10%, and 100% of the intensity of the detected signals, respectively. We first show its influence on the MS scheme. As expected, the separation performance gets worse as the noise intensity increases, as shown in Fig. 15.

Next, let us treat SPARS. For a typical example, we take the reference signal as $23\sin(17t)$. As expected, the noise affects the separation performance of SPARS. Higher noise intensity causes a worse source mismatch, though the reference mismatch and the error function are almost the same, as shown in Fig. 16. Again, it is worthwhile to note that the reference amplitude has no influence on limit of source mismatch. As an example, we take reference signal $230\sin(17t)$

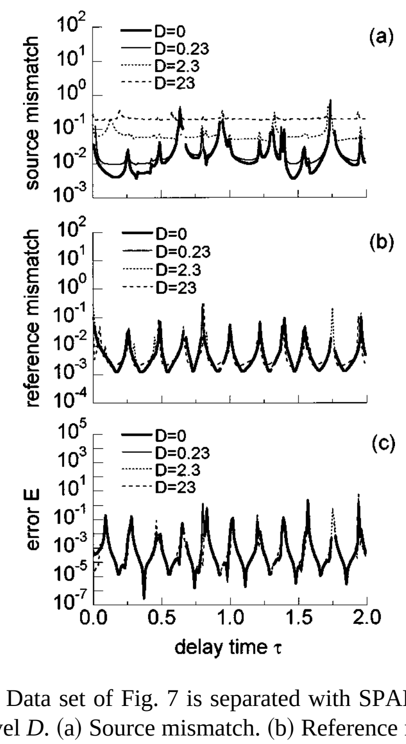

FIG. 16. Data set of Fig. 7 is separated with SPARS for different noise level *D*. (a) Source mismatch. (b) Reference mismatch. (c) Error function *E*.

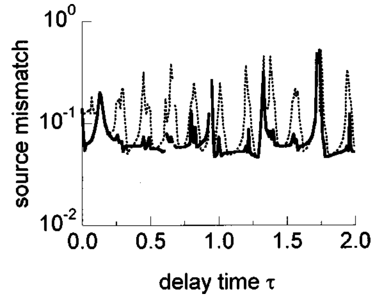

FIG. 17. Data set of Fig. 7 is separated with SPARS for the same noise level $D=2.3$. Solid line is the performance of reference signal $23\sin(17t)$. Dashed line is the performance of reference signal $230\sin(17t)$.

(dashed line) and recalculate the separation performance of SPARS. The result is shown in Fig. 17. The lower bounds of source mismatch are almost the same.

IX. CONCLUSIONS

The blind source separation based on the MS scheme is physically appealing and the whole problem can be reduced to an eigenvector problem elegantly. However, in practice two important factors—source cross-correlation and data length—can seriously hinder the separation performance in the MS method and they were left unsolved. We emphasize that finite source cross-correlation is inevitable when the spectra of the sources are overlapping and this source cross dependence is hardly justified in advance. In the meantime, the data length in practical analysis is always finite. Although these difficulties also occur with the other methods of blind source separation [4,10], extensive development and implementation has been achieved for these noneigenvalue problem schemes [4,10]. In contrast, to our knowledge, no practical implementations and/or devices have been developed for the MS scheme [15]. As seen above, even with finite data length and source cross-correlation, the separation can still be carried out for the MS scheme. But the problem is that there are too many available time delays τ for separation. This turns out to lead to a serious optimization problem. This problem actually had remained an unsolved difficulty in applications with the MS scheme. We should mention that in the work of Molgedey and Schuster, they had introduced an idea of cost function [1]. This cost function approach can be combined with the eigenvalue problem approach for solving the separation problem with correlated signals as shown in Ref. [12].

As shown above, with the use of a carefully chosen reference signal, one can find an operationally available τ to separate the mixed signals with an optimum minimum mismatch for the reconstructed signals and the original sources. It should be emphasized that the eigenvalue problem approach is still without any cost function. We have provided a heuristic illustration to show its reliability. Furthermore, our simulations strongly support its extension to different models and larger system size , though we do not present the numerical details here. The influences of nonzero cross-correlation, finite data length, and channel noise on signal separation have been extensively analyzed. It has been shown that in the worst case all these factors—cross-correlation, data length, and channel noise—can destroy the possibility of separation and result in failure. On the other hand, with finite data length, if the cross-correlation is not so strong and the channel noise is small, the separation problem will lead to an optimization problem, and it is solvable by the SPARS method as shown above. Nevertheless, we should report our results of the strong correlation case, although, in strong correlation, the separation performance of SPARS is better than that of the MS scheme. It turns out that the performance is poor, unfortunately.

Essentially, our result shows that if we can tune the frequency such that $a'_{\rm ref}\rightarrow a_{\rm ref}$, $A'\rightarrow S^{-1}A$ automatically holds after solving for the eigenvectors. Is it possible $A'\rightarrow S^{-1}A \rightarrow A+O(\varepsilon^2)$ or *even better* for the chosen τ? Our analysis shows that a perfectly noncorrelated reference signal does not provide this improvement. However, with a reference signal that has the least correlation to the detected signals, improvement is possible. With this approach it is possible to achieve a better separation performance in comparison with the MS scheme, particularly in the case where source cross-correlation is not weak. At this stage, the optimum separation performance achieved by SPARS is only with numerical supports. Nevertheless, it is worthwhile to call attention to this intuitive approach. This approach has pointed out a possible path toward a better solution in the field of blind signal separation. It is worthwhile to emphasize that practical implementation of optimization is straightforward. The optimization of blind source separation demonstrated here should have a rather positive meaning. Extensions to cases where the number of sources is unknown and experimental implementation to optical signal separation where interference may be important are currently in progress.

ACKNOWLEDGMENT

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[1] J. Maddox, Nature (London) **369**, 517 (1994).
[2] H. W. Strube, Signal Process. **3**, 355 (1981).
[3] C. M. Gray, A. K. Engel, W. Singer, and P. Konig, Nature (London) **338**, 334 (1989).
[4] C. Jutten and J. Herault, Signal Process. **24**, 1 (1991).
[5] J. J. Hopfield, Proc. Natl. Acad. Sci. USA **88**, 6462 (1991).
[6] A. Cichocki and R. Unbehauen, *Neural Networks for Optimization and Signal Processing* (Wiley, New York, 1995).
[7] A. J. Bell and T. J. Sejnowski, Neural Comput. **7**, 1129 (1995).
[8] P. Comon, Signal Process. **36**, 287 (1994).
[9] D. Yellin and E. Weinstein, IEEE Trans. Signal Process. **42**, 2158 (1994).
[10] S. Haykin, *Blind Deconvolution* (Prentice-Hall, Englewood Cliffs, NJ, 1994).
[11] L. Molgedey and H. G. Schuster, Phys. Rev. Lett. **72**, 3634 (1994).
[12] F. Ehlers and H. G. Schuster, IEEE Trans. Signal Process. **45**, 2608 (1997).
[13] D. N. Ghosh Roy, *Methods of Inverse Problems in Physics* (CRC, Cleveland, 1991).
[14] H. G. Schuster, *Deterministic Chaos*, 3rd ed. (VCH, Weinheim, 1995).
[15] We check the available SCI (science citation index) data set up to September 1997. Only two papers, Refs. [1] and [7], cite Molgedey and Schuster's paper. On the other hand, there have been over 45 papers citing the work of Jutten and Herault, including a variety of practical devices and patents.